

The entrainment matrix of a superfluid neutron-proton mixture at a finite temperature

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Abstract

The entrainment matrix (also termed the Andreev-Bashkin matrix or the mass-density matrix) for a neutron-proton mixture is derived at a finite temperature in a neutron star core. The calculation is performed in the frame of the Landau Fermi-liquid theory generalized to account for superfluidity of nucleons. It is shown, that the temperature dependence of the entrainment matrix is described by a universal function independent on an actual model of nucleon-nucleon interaction employed. The results are presented in the form convenient for their practical use. The entrainment matrix is important, e.g., in kinetics of superfluid nucleon mixtures or in studies of the dynamical evolution of neutron stars (in particular, in the studies of star pulsations and pulsar glitches).

PACS: 21.65.+f; 71.10.Ay; 97.60.Jd; 26.60.+c

Keywords: Neutron star matter; Fermi liquid theory; Superfluidity; Nucleon superfluid densities

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1 Introduction

It is well known, that the neutron star core becomes superfluid (superconducting) at a certain stage of neutron star thermal evolution (see, e.g., Ref. [1]). It is generally agreed that protons pair in the spin singlet (1S_0) state, while neutrons pair in the spin triplet (3P_2) state in the neutron star core. A variety of different models of nucleon pairing have been proposed in literature (references to original papers can be found in Yakovlev et al. [2] and in Lombardo and Schulze [1]). These models predict very different density profiles of neutron (n) or proton (p) critical temperatures $T_{\text{cn,p}}(\rho)$. In addition, it is not absolutely clear whether the actual projection of angular momentum of neutron pair onto quantization axis is $m_J = 0$ (as one usually assumes) or it can be $|m_J| = 1$ or 2. For example, Amundsen and Østgaard [3] found that the energetically preferable state of a neutron pair can be a superposition of states with different m_J .

Despite many theoretical uncertainties it is obvious that superfluidity strongly affects the evolution of neutron stars, for example, its cooling (see, e.g., Ref. [4]), neutron star pulsations (see, e.g., Refs. [5] – [10]) and is probably related to pulsar glitches (see Refs. [11,12]).

One of the key ingredients of hydrodynamics and kinetics of superfluid mixtures is the *entrainment matrix* $\rho_{\alpha\alpha'}$ (also termed the Andreev-Bashkin matrix or the mass-density matrix). Implying, for simplicity, that the only baryons in a neutron star core are neutrons and protons, the matrix $\rho_{\alpha\alpha'}$ can be defined as [13]:

$$\mathbf{J}_n = (\rho_n - \rho_{nn} - \rho_{np}) \mathbf{V}_{qp} + \rho_{nn} \mathbf{V}_{ns} + \rho_{np} \mathbf{V}_{ps}, \quad (1)$$

$$\mathbf{J}_p = (\rho_p - \rho_{pp} - \rho_{pn}) \mathbf{V}_{qp} + \rho_{pp} \mathbf{V}_{ps} + \rho_{pn} \mathbf{V}_{ns}. \quad (2)$$

Here $\rho_\alpha = m_\alpha n_\alpha$; n_α and m_α are the number density and the mass of nucleon species $\alpha = n$ or p ; \mathbf{J}_α and $\mathbf{V}_{\alpha s}$ are the mass current density and the superfluid velocity; \mathbf{V}_{qp} is the normal velocity of thermal excitations (see, e.g., Refs. [14,15]). We assume that \mathbf{V}_{qp} is the same for nucleons of both species. Eqs. (1) and (2) differ from a “natural” expression for the mass current density $\mathbf{J}_\alpha = \rho_\alpha \mathbf{V}_\alpha$ (with \mathbf{V}_α being the momentum per unit mass of nucleon species α) for two reasons.

First, three independent motions can exist in a mixture of two superfluids, each carrying a mass (see, e.g., Ref. [16]). They are the motion of thermal excitations with the velocity \mathbf{V}_{qp} and two superfluid motions with the velocities \mathbf{V}_{ns} and \mathbf{V}_{ps} .

Second, the superfluid flow of one component of the mixture entrains a flow

of another component and vice versa. For example, the superfluid motion of neutrons carries along some part of the mass of protons because nucleon liquids are strongly interacting (see, e.g., Ref. [17]). The non-diagonal elements ρ_{np} and ρ_{pn} , therefore, characterize the intensity of neutron-proton coupling. In particular, in the absence of interactions between neutrons and protons one has $\rho_{np} = \rho_{pn} = 0$.

It follows from the phenomenological analysis of Andreev and Bashkin [13] that the entrainment matrix should be symmetric ($\rho_{np} = \rho_{pn}$), since it can be presented in the form:

$$\rho_{\alpha\alpha'} = \left(\frac{\partial^2 E}{\partial \mathbf{V}_{\alpha s} \partial \mathbf{V}_{\alpha' s}} \right)_{n_n, n_p, T}, \quad (3)$$

where E is the energy density in the coordinate frame in which $\mathbf{V}_{qp} = 0$, and T is the temperature.

The entrainment matrix $\rho_{\alpha\alpha'}$ of a neutron-proton mixture was calculated by Borumand et al. [18] and Comer and Joynt [19] for $T = 0$. However, in many cases the zero-temperature approximation cannot be justified. For example, one needs the matrix $\rho_{\alpha\alpha'}$ at non-zero temperatures for analyzing kinetic properties of matter, especially the kinetic coefficients (the bulk and shear viscosity, the diffusion coefficient). Also, the entrainment matrix is required at $T \neq 0$ for investigating pulsations of warm neutron stars or stars possessing the pulsation energy of the order of or higher than its thermal energy (It is implied that pulsation energy, being dissipated, is able to heat the star substantially and thus to change $\rho_{\alpha\alpha'}$). A simple example, illustrating the influence of temperature related effects on the neutron star pulsations, was considered by Gusakov et al. [20]).

In this paper the entrainment matrix of a neutron-proton mixture is derived for non-zero temperatures. Calculations are performed in the frame of the Landau Fermi-liquid theory, generalized by Larkin and Migdal [21] and Leggett [22] to the case of superfluidity. For the sake of simplicity we assume the singlet-state (1S_0) pairing of nucleons of both species. In Section 4.1 we will show how the obtained results can be extended to the case of triplet-state 3P_2 neutron pairing.

2 A current-free neutron-proton mixture

Before calculating the matrix $\rho_{\alpha\alpha'}$, let us consider a neutron-proton mixture in the absence of currents. A simple generalization of the Hamiltonian of a superfluid Fermi-liquid, suggested by Leggett [22], to the case of superfluid

mixtures, gives

$$H - \mu_n N_n - \mu_p N_p = H_{\text{LF}} + H_{\text{pairing}}. \quad (4)$$

Here, H is the hamiltonian of the system which is the sum of the standard Fermi-liquid hamiltonian H_{LF} for mixtures and the pairing hamiltonian H_{pairing} ; N_α and μ_α are the number density operator and the chemical potential of nucleon species α , respectively. The expression for H_{LF} has the form (see, e.g., Ref. [17]):

$$\begin{aligned} H_{\text{LF}} = & \sum_{\mathbf{p}\sigma\alpha} \varepsilon_0^{(\alpha)}(\mathbf{p}) \left(a_{\mathbf{p}}^{(\alpha)\dagger} a_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)} \right) \\ & + \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'\alpha\alpha'} f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}') \left(a_{\mathbf{p}}^{(\alpha)\dagger} a_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)} \right) \left(a_{\mathbf{p}'}^{(\alpha')\dagger} a_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')} \right). \end{aligned} \quad (5)$$

In Eq. (5) the summation is taken over the particle momenta \mathbf{p} and \mathbf{p}' , as well as over the spin projections σ and σ' onto the quantization axis and over the particle species $\alpha, \alpha' = n$ or p ; $a_{\mathbf{p}}^{(\alpha)} \equiv a_{\mathbf{p}\sigma}^{(\alpha)} = a_{\mathbf{p}\uparrow}^{(\alpha)}$ or $a_{\mathbf{p}\downarrow}^{(\alpha)}$ is the annihilation operator of a quasiparticle (not the Bogoliubov excitation!) of species α in a state $(\mathbf{p}\sigma)$. We restrict ourselves to a spin-unpolarized nucleon matter. This allows us to simplify the notations. For instance, we will drop spin indices, whenever possible. Furthermore, $\theta_{\mathbf{p}}^{(\alpha)} = \theta(p_{\text{F}\alpha} - |\mathbf{p}|)$, where $\theta(x)$ is the step function; $\varepsilon_0^{(\alpha)}(\mathbf{p}) = v_{\text{F}\alpha}(|\mathbf{p}| - p_{\text{F}\alpha})$, where $v_{\text{F}\alpha}$ and $p_{\text{F}\alpha}$ are, respectively, the Fermi-velocity and Fermi-momentum; $f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')$ is the spin-averaged Landau quasiparticle interaction (we disregard the spin-dependence of this interaction since it does not affect our results); $f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}')$ is the (spin-averaged) second variational derivative of the energy with respect to the number of particles. Therefore, it is invariant under transformations $\mathbf{p} \rightleftharpoons \mathbf{p}'$ and $\alpha \rightleftharpoons \alpha'$ (see, e.g., Refs. [14,17]):

$$f^{\alpha\alpha'}(\mathbf{p},\mathbf{p}') = f^{\alpha'\alpha}(\mathbf{p},\mathbf{p}') = f^{\alpha\alpha'}(\mathbf{p}',\mathbf{p}). \quad (6)$$

The pairing hamiltonian can be written as:

$$H_{\text{pairing}} = \sum_{\mathbf{p}\mathbf{p}'\alpha} \mathcal{V}^{(\alpha)}(\mathbf{p},\mathbf{p}') a_{\mathbf{p}\uparrow}^{(\alpha)\dagger} a_{-\mathbf{p}'\downarrow}^{(\alpha)\dagger} a_{-\mathbf{p}\downarrow}^{(\alpha)} a_{\mathbf{p}'\uparrow}^{(\alpha)}. \quad (7)$$

Here we assume the following symmetry conditions:

$$\mathcal{V}^{(\alpha)}(\mathbf{p},\mathbf{p}') = \mathcal{V}^{(\alpha)}(\mathbf{p}',\mathbf{p}) = \mathcal{V}^{(\alpha)}(\mathbf{p},-\mathbf{p}') = \mathcal{V}^{(\alpha)}(-\mathbf{p},\mathbf{p}'). \quad (8)$$

If the matrix element $\mathcal{V}^{(\alpha)}(\mathbf{p},\mathbf{p}')$ does not satisfy these conditions (e.g., for singlet-state pairing of nucleon pairs with non-zero orbital angular momentum), it should be symmetrized in such a way as to obey Eq. (8) (see, e.g.,

Ref. [23]). To diagonalize the operator (4) we introduce the standard operators of Bogoliubov excitations $b_{\mathbf{p}\sigma}^{(\alpha)}$:

$$a_{\mathbf{p}\uparrow}^{(\alpha)} = u_{\mathbf{p}}^{(\alpha)} b_{\mathbf{p}\uparrow}^{(\alpha)} + v_{\mathbf{p}}^{(\alpha)} b_{-\mathbf{p}\downarrow}^{(\alpha)\dagger}, \quad (9)$$

$$a_{\mathbf{p}\downarrow}^{(\alpha)} = u_{\mathbf{p}}^{(\alpha)} b_{\mathbf{p}\downarrow}^{(\alpha)} - v_{\mathbf{p}}^{(\alpha)} b_{-\mathbf{p}\uparrow}^{(\alpha)\dagger}. \quad (10)$$

The parameters $u_{\mathbf{p}}^{(\alpha)}$ and $v_{\mathbf{p}}^{(\alpha)}$ are related by the inversion symmetry and normalization

$$u_{\mathbf{p}}^{(\alpha)} = u_{-\mathbf{p}}^{(\alpha)}, \quad v_{\mathbf{p}}^{(\alpha)} = v_{-\mathbf{p}}^{(\alpha)}, \quad u_{\mathbf{p}}^{(\alpha)2} + v_{\mathbf{p}}^{(\alpha)2} = 1. \quad (11)$$

Now one can find the eigenvalues of the operator (4)

$$\begin{aligned} E - \mu_n n_n - \mu_p n_p &= \sum_{\mathbf{p}\sigma\alpha} \varepsilon_0^{(\alpha)}(\mathbf{p}) \left(n_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)} \right) \\ &+ \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'\alpha\alpha'} f^{\alpha\alpha'}(\mathbf{p}, \mathbf{p}') \left(n_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)} \right) \left(n_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')} \right) \\ &+ \sum_{\mathbf{p}\mathbf{p}'\alpha} \mathcal{V}^{(\alpha)}(\mathbf{p}, \mathbf{p}') u_{\mathbf{p}}^{(\alpha)} v_{\mathbf{p}}^{(\alpha)} u_{\mathbf{p}'}^{(\alpha)} v_{\mathbf{p}'}^{(\alpha)} \left(1 - 2f_{\mathbf{p}}^{(\alpha)} \right) \left(1 - 2f_{\mathbf{p}'}^{(\alpha)} \right), \end{aligned} \quad (12)$$

where $n_{\mathbf{p}}^{(\alpha)}$ and $f_{\mathbf{p}}^{(\alpha)}$ are the distribution functions of quasiparticles and Bogoliubov excitations, respectively, given by

$$n_{\mathbf{p}}^{(\alpha)} = \langle |a_{\mathbf{p}\uparrow}^{(\alpha)\dagger} a_{\mathbf{p}\uparrow}^{(\alpha)}| \rangle = \langle |a_{\mathbf{p}\downarrow}^{(\alpha)\dagger} a_{\mathbf{p}\downarrow}^{(\alpha)}| \rangle = v_{\mathbf{p}}^{(\alpha)2} + \left(u_{\mathbf{p}}^{(\alpha)2} - v_{\mathbf{p}}^{(\alpha)2} \right) f_{\mathbf{p}}^{(\alpha)}, \quad (13)$$

$$f_{\mathbf{p}}^{(\alpha)} = \langle |b_{\mathbf{p}\uparrow}^{(\alpha)\dagger} b_{\mathbf{p}\uparrow}^{(\alpha)}| \rangle = \langle |b_{\mathbf{p}\downarrow}^{(\alpha)\dagger} b_{\mathbf{p}\downarrow}^{(\alpha)}| \rangle. \quad (14)$$

The entropy of the system is given by the usual combinatorial expression:

$$S = - \sum_{\mathbf{p}\sigma\alpha} \left[\left(1 - f_{\mathbf{p}}^{(\alpha)} \right) \ln \left(1 - f_{\mathbf{p}}^{(\alpha)} \right) + f_{\mathbf{p}}^{(\alpha)} \ln f_{\mathbf{p}}^{(\alpha)} \right]. \quad (15)$$

Minimizing the thermodynamical potential $F = E - \mu_n n_n - \mu_p n_p - TS$ with respect to unknown functions $f_{\mathbf{p}}^{(\alpha)}$ and $u_{\mathbf{p}}^{(\alpha)}$ ($v_{\mathbf{p}}^{(\alpha)}$ can be expressed through $u_{\mathbf{p}}^{(\alpha)}$ using Eq. (11)), one finds

$$f_{\mathbf{p}}^{(\alpha)} = \frac{1}{1 + e^{E_{\mathbf{p}}^{(\alpha)}/T}}, \quad E_{\mathbf{p}}^{(\alpha)} = \sqrt{\varepsilon^{(\alpha)2}(\mathbf{p}) + \Delta_{\mathbf{p}}^{(\alpha)2}}, \quad (16)$$

$$u_{\mathbf{p}}^{(\alpha)2} = \frac{1}{2} \left(1 + \frac{\varepsilon^{(\alpha)}(\mathbf{p})}{E_{\mathbf{p}}^{(\alpha)}} \right). \quad (17)$$

Here and below we use the system of units in which $\hbar = k_B = V = 1$, where \hbar is the Planck constant, k_B is the Boltzmann constant, and V is the normalization volume. In Eqs. (16) – (17) $E_{\mathbf{p}}^{(\alpha)}$ is the energy of a Bogoliubov excitation with

momentum \mathbf{p} . The superfluid gap $\Delta_{\mathbf{p}}^{(\alpha)}$ of nucleon species α can be determined from the equation

$$\Delta_{\mathbf{p}}^{(\alpha)} = - \sum_{\mathbf{p}'} \mathcal{V}^{(\alpha)}(\mathbf{p}, \mathbf{p}') u_{\mathbf{p}'}^{(\alpha)} v_{\mathbf{p}'}^{(\alpha)} \left(1 - 2f_{\mathbf{p}'}^{(\alpha)}\right). \quad (18)$$

Finally, $\varepsilon^{(\alpha)}(\mathbf{p})$ is the quantity which formally coincides with the energy of quasiparticle species α in the mixture of non-superfluid Fermi-liquids,

$$\varepsilon^{(\alpha)}(\mathbf{p}) = \varepsilon_0^{(\alpha)}(\mathbf{p}) + \sum_{\mathbf{p}'\sigma'\alpha'} f^{\alpha\alpha'}(\mathbf{p}, \mathbf{p}') \left(n_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')}\right). \quad (19)$$

In Eq. (19) the quasiparticle distribution function $n_{\mathbf{p}}^{(\alpha)}$ is determined by Eq. (13) with $f_{\mathbf{p}}^{(\alpha)}$ and $u_{\mathbf{p}}^{(\alpha)}$ taken from Eqs. (16)–(17). The first term in the right-hand side of Eq. (19) can be estimated as: $\varepsilon_0^{(\alpha)}(\mathbf{p}) \sim (T + \Delta^{(\alpha)})$, where $\Delta^{(\alpha)}$ is a typical value of the gap. In thermodynamic equilibrium, the second term in Eq. (19) is much smaller than the first one because for any function $f(p)$, smooth in the vicinity of a Fermi surface, one has the estimate:

$$\int_0^\infty f(p) p^2 (n_{\mathbf{p}}^{(\alpha)} - \theta_{\mathbf{p}}^{(\alpha)}) dp \sim f(p_{\text{F}\alpha}) n_{\alpha} \left([T/\mu_{\alpha}]^2 + [\Delta^{(\alpha)}/\mu_{\alpha}]^2\right), \quad (20)$$

Thus, since $(T + \Delta^{(\alpha)})/\mu_{\alpha} \ll 1$, the second term in Eq. (19) can be neglected.

3 A neutron-proton mixture with superfluid currents

3.1 General consideration

In a system with superfluid currents the plane-wave states of nucleons $(\mathbf{p} + \mathbf{Q}_{\alpha}, \uparrow)$ and $(-\mathbf{p} + \mathbf{Q}_{\alpha}, \downarrow)$ are paired (note, that we consider singlet-state pairing of both species and assume $\mathbf{Q}_{\alpha} \equiv m_{\alpha} \mathbf{V}_{\alpha s} \ll p_{\text{F}\alpha}$). In this case the pairing hamiltonian should be written as (see, e.g., Refs. [24,25])

$$H_{\text{pairing}}(\mathbf{Q}_{\alpha}) = \sum_{\mathbf{p}\mathbf{p}'\alpha} \mathcal{V}_{\mathbf{Q}_{\alpha}}^{(\alpha)}(\mathbf{p}, \mathbf{p}') a_{\mathbf{p}'+\mathbf{Q}_{\alpha}\uparrow}^{(\alpha)\dagger} a_{-\mathbf{p}'+\mathbf{Q}_{\alpha}\downarrow}^{(\alpha)\dagger} a_{-\mathbf{p}+\mathbf{Q}_{\alpha}\downarrow}^{(\alpha)} a_{\mathbf{p}+\mathbf{Q}_{\alpha}\uparrow}^{(\alpha)}. \quad (21)$$

Here $\mathcal{V}_{\mathbf{Q}_{\alpha}}^{(\alpha)}(\mathbf{p}, \mathbf{p}')$ is the matrix element for the scattering of a pair of quasiparticles species α from states $(\mathbf{p} + \mathbf{Q}_{\alpha}, \uparrow)$, $(-\mathbf{p} + \mathbf{Q}_{\alpha}, \downarrow)$ to states $(\mathbf{p}' + \mathbf{Q}_{\alpha}, \uparrow)$, $(-\mathbf{p}' + \mathbf{Q}_{\alpha}, \downarrow)$. Due to the rotational invariance, the expansion of $\mathcal{V}_{\mathbf{Q}_{\alpha}}^{(\alpha)}(\mathbf{p}, \mathbf{p}')$ in powers of \mathbf{Q}_{α} will contain the terms $\sim Q_{\alpha}^2$ and higher. Therefore, as we will work in the linear approximation in \mathbf{Q}_{α} , we will neglect the dependence of $\mathcal{V}_{\mathbf{Q}_{\alpha}}^{(\alpha)}(\mathbf{p}, \mathbf{p}')$ on momentum \mathbf{Q}_{α} and put $\mathcal{V}_{\mathbf{Q}_{\alpha}}^{(\alpha)}(\mathbf{p}, \mathbf{p}') \approx \mathcal{V}^{(\alpha)}(\mathbf{p}, \mathbf{p}')$.

Expressing the quasiparticle operators in terms of Bogoliubov excitation operators

$$a_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)} = U_{\mathbf{p}}^{(\alpha)} b_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)} + V_{\mathbf{p}}^{(\alpha)} b_{-\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)\dagger}, \quad (22)$$

$$a_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)} = U_{\mathbf{p}}^{(\alpha)} b_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)} - V_{\mathbf{p}}^{(\alpha)} b_{-\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)\dagger}, \quad (23)$$

where $U_{\mathbf{p}}^{(\alpha)}$ and $V_{\mathbf{p}}^{(\alpha)}$ satisfy the equalities

$$U_{\mathbf{p}}^{(\alpha)} = U_{-\mathbf{p}}^{(\alpha)}, \quad V_{\mathbf{p}}^{(\alpha)} = V_{-\mathbf{p}}^{(\alpha)}, \quad U_{\mathbf{p}}^{(\alpha)2} + V_{\mathbf{p}}^{(\alpha)2} = 1, \quad (24)$$

similar to (11), one obtains the expression for the energy density:

$$\begin{aligned} E - \mu_n n_n - \mu_p n_p &= \sum_{\mathbf{p}\sigma\alpha} \varepsilon_0^{(\alpha)}(\mathbf{p} + \mathbf{Q}_\alpha) \left(\mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - \theta_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \right) \\ &+ \frac{1}{2} \sum_{\mathbf{p}\mathbf{p}'\sigma\sigma'\alpha\alpha'} f^{\alpha\alpha'}(\mathbf{p} + \mathbf{Q}_\alpha, \mathbf{p}' + \mathbf{Q}_{\alpha'}) \left(\mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - \theta_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \right) \left(\mathcal{N}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}^{(\alpha')} - \theta_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}^{(\alpha')} \right) \\ &\quad + \sum_{\mathbf{p}\mathbf{p}'\alpha} \mathcal{V}^{(\alpha)}(\mathbf{p}, \mathbf{p}') U_{\mathbf{p}}^{(\alpha)} V_{\mathbf{p}}^{(\alpha)} U_{\mathbf{p}'}^{(\alpha)} V_{\mathbf{p}'}^{(\alpha)} \\ &\quad \times \left(1 - \mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - \mathcal{F}_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \right) \left(1 - \mathcal{F}_{\mathbf{p}'+\mathbf{Q}_\alpha}^{(\alpha)} - \mathcal{F}_{-\mathbf{p}'+\mathbf{Q}_\alpha}^{(\alpha)} \right). \end{aligned} \quad (25)$$

Here $\mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ and $\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ are the distribution functions of quasiparticles and Bogoliubov excitations with momentum $(\mathbf{p} + \mathbf{Q}_\alpha)$, respectively:

$$\begin{aligned} \mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} &= \langle |a_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)\dagger} a_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)}| \rangle = \langle |a_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)\dagger} a_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)}| \rangle \\ &= V_{\mathbf{p}}^{(\alpha)2} + U_{\mathbf{p}}^{(\alpha)2} \mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - V_{\mathbf{p}}^{(\alpha)2} \mathcal{F}_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}, \end{aligned} \quad (26)$$

$$\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \langle |b_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)\dagger} b_{\mathbf{p}+\mathbf{Q}_\alpha\uparrow}^{(\alpha)}| \rangle = \langle |b_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)\dagger} b_{\mathbf{p}+\mathbf{Q}_\alpha\downarrow}^{(\alpha)}| \rangle. \quad (27)$$

The entropy of the system is still given by Eq. (15) with the distribution function $\mathfrak{f}_{\mathbf{p}}^{(\alpha)}$ replaced by $\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$. The minimization of the thermodynamical potential $F = E - \mu_n n_n - \mu_p n_p - TS$ with respect to $\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ and $U_{\mathbf{p}}^{(\alpha)}$ yields

$$\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \frac{1}{1 + e^{\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}/T}}, \quad (28)$$

$$\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \frac{1}{2} \left(H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - H_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \right) + \sqrt{\frac{1}{4} \left(H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} + H_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \right)^2 + \mathcal{D}_{\mathbf{p}}^{(\alpha)2}}, \quad (29)$$

$$U_{\mathbf{p}}^{(\alpha)2} = \frac{1}{2} \left(1 + \frac{H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} + H_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{2\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} + H_{-\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} - H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}} \right). \quad (30)$$

In Eqs. (28)–(30) $\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ is the energy of a Bogoliubov excitation with momentum $(\mathbf{p} + \mathbf{Q}_\alpha)$. The stability of the system implies $\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} \geq 0$. Furthermore,

$\mathcal{D}_{\mathbf{p}}^{(\alpha)}$ is the superfluid gap which can be found from the equation:

$$\mathcal{D}_{\mathbf{p}}^{(\alpha)} = - \sum_{\mathbf{p}'} \mathcal{V}^{(\alpha)}(\mathbf{p}, \mathbf{p}') U_{\mathbf{p}'}^{(\alpha)} V_{\mathbf{p}'}^{(\alpha)} \left(1 - \mathcal{F}_{\mathbf{p}'+\mathbf{Q}_\alpha}^{(\alpha)} - \mathcal{F}_{-\mathbf{p}'+\mathbf{Q}_\alpha}^{(\alpha)} \right). \quad (31)$$

Finally, $H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ is the quantity which formally coincides with the energy of a quasiparticle with momentum $(\mathbf{p} + \mathbf{Q}_\alpha)$ in the mixture of non-superfluid Fermi-liquids,

$$H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \varepsilon_0^{(\alpha)}(\mathbf{p} + \mathbf{Q}_\alpha) + \sum_{\mathbf{p}'\sigma'\alpha'} f^{\alpha\alpha'}(\mathbf{p} + \mathbf{Q}_\alpha, \mathbf{p}' + \mathbf{Q}_{\alpha'}) \left(\mathcal{N}_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}^{(\alpha')} - \theta_{\mathbf{p}'+\mathbf{Q}_{\alpha'}}^{(\alpha')} \right). \quad (32)$$

The quasiparticle distribution function $\mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ is defined by Eq. (26). Since $Q_\alpha \ll p_{\text{F}\alpha}$, one can expand $H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$ in terms of $Q_{\alpha'}$ ($\alpha' = \text{n, p}$) and write

$$H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \varepsilon^{(\alpha)}(\mathbf{p}) + \Delta H_{\mathbf{p}}^{(\alpha)}. \quad (33)$$

In the case of singlet-state nucleon pairing there are only three vectors \mathbf{p} , \mathbf{Q}_n and \mathbf{Q}_p which can form the scalar $\Delta H_{\mathbf{p}}^{(\alpha)}$. Neglecting all terms which are quadratic and higher order in $Q_\alpha/p_{\text{F}\alpha}$, one can write

$$\Delta H_{\mathbf{p}}^{(\alpha)} = \sum_{\alpha'} \gamma_{\alpha\alpha'}(\mathbf{p}) \frac{\mathbf{p} \mathbf{Q}_{\alpha'}}{m_{\alpha'}}, \quad (34)$$

where $\gamma_{\alpha\alpha'}(\mathbf{p})$ is the matrix to be derived in the next section on the Fermi surface of particle species α (at $p = p_{\text{F}\alpha}$). Taking into account Eqs. (8), (29)–(31), and (33)–(34) and neglecting the terms $\sim Q_\alpha^2$, one has

$$\mathcal{D}_{\mathbf{p}}^{(\alpha)} = \Delta_{\mathbf{p}}^{(\alpha)}, \quad U_{\mathbf{p}}^{(\alpha)} = u_{\mathbf{p}}^{(\alpha)}, \quad V_{\mathbf{p}}^{(\alpha)} = v_{\mathbf{p}}^{(\alpha)}. \quad (35)$$

Now the energy of Bogoliubov excitations $\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}$, as well as the distribution functions of quasiparticles and Bogoliubov excitations can be expanded in analogy with Eq. (33) as

$$\mathfrak{E}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = E_{\mathbf{p}}^{(\alpha)} + \Delta H_{\mathbf{p}}^{(\alpha)}, \quad (36)$$

$$\mathcal{F}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = \mathfrak{f}_{\mathbf{p}}^{(\alpha)} + \frac{\partial \mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}} \Delta H_{\mathbf{p}}^{(\alpha)}, \quad \mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)} = n_{\mathbf{p}}^{(\alpha)} + \frac{\partial \mathfrak{f}_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}} \Delta H_{\mathbf{p}}^{(\alpha)}. \quad (37)$$

3.2 The calculation of matrix $\gamma_{\alpha\alpha'}$

To calculate the matrix $\gamma_{\alpha\alpha'}(p_{\text{F}\alpha})$ we will make use of Eq. (32). Restricting ourselves to the terms linear in \mathbf{Q}_α , we expand all functions in Eq. (32) using

the formulas (33) and (37). Then, taking into account Eqs. (19) and (20), and neglecting all terms in Eq. (32) which depend on $(n_{\mathbf{p}'}^{(\alpha')} - \theta_{\mathbf{p}'}^{(\alpha')})$, we obtain

$$\Delta H_{\mathbf{p}}^{(\alpha)} = \frac{\mathbf{p} \mathbf{Q}_{\alpha}}{m_{\alpha}^*} + \sum_{\mathbf{p}' \sigma' \alpha'} f^{\alpha \alpha'}(\mathbf{p}, \mathbf{p}') \left\{ \frac{\partial f_{\mathbf{p}'}^{(\alpha')}}{\partial E_{\mathbf{p}'}^{(\alpha')}} \Delta H_{\mathbf{p}'}^{(\alpha')} - \frac{\partial \theta_{\mathbf{p}'}^{(\alpha')}}{\partial \mathbf{p}'} \mathbf{Q}_{\alpha'} \right\}. \quad (38)$$

Let us calculate the sum in this equation. The main contribution to the sum comes from a narrow region of $|\mathbf{p}'| \sim p_{F\alpha'}$ since the function in the curly brackets is essentially non-zero only close to the Fermi surface of particle species α' . Furthermore, in a smooth function $f^{\alpha \alpha'}(\mathbf{p}, \mathbf{p}')$ we replace $|\mathbf{p}|$ and $|\mathbf{p}'|$ by $p_{F\alpha}$ and $p_{F\alpha'}$, respectively, and expand it into Legendre polynomials $P_l(\cos \theta)$:

$$f^{\alpha \alpha'}(\mathbf{p}, \mathbf{p}') = \sum_l f_l^{\alpha \alpha'} P_l(\cos \theta), \quad (39)$$

where θ is the angle between \mathbf{p} and \mathbf{p}' . Using isotropy of the gaps $\Delta_{\mathbf{p}}^{(\alpha)}$ ($\alpha = n, p$) and Eq. (34) we obtain

$$\sum_{\mathbf{p}' \sigma'} f^{\alpha \alpha'}(\mathbf{p}, \mathbf{p}') \frac{\partial f_{\mathbf{p}'}^{(\alpha')}}{\partial E_{\mathbf{p}'}^{(\alpha')}} \Delta H_{\mathbf{p}'}^{(\alpha')} = - \frac{f_1^{\alpha \alpha'} N_{0\alpha'}}{3} \frac{p_{F\alpha'}}{p_{F\alpha}} \Phi_{\alpha'} \Delta H_{\mathbf{p}}^{(\alpha')}, \quad (40)$$

$$\sum_{\mathbf{p}' \sigma'} f^{\alpha \alpha'}(\mathbf{p}, \mathbf{p}') \frac{\partial \theta_{\mathbf{p}'}^{(\alpha')}}{\partial \mathbf{p}'} \mathbf{Q}_{\alpha'} = - \frac{f_1^{\alpha \alpha'} N_{0\alpha'}}{3} \frac{p_{F\alpha'}}{p_{F\alpha}} \frac{\mathbf{p} \mathbf{Q}_{\alpha'}}{m_{\alpha'}^*}. \quad (41)$$

Here $N_{0\alpha} = m_{\alpha}^* p_{F\alpha} / \pi^2$; $m_{\alpha}^* = p_{F\alpha} / v_{F\alpha}$ is the effective mass of particle species α . The function Φ_{α} , which is given by

$$\Phi_{\alpha} = - \frac{1}{N_{0\alpha}} \sum_{\mathbf{p} \sigma} \frac{\partial f_{\mathbf{p}}^{(\alpha)}}{\partial E_{\mathbf{p}}^{(\alpha)}}, \quad (42)$$

was calculated numerically and approximated by Gnedin and Yakovlev [26]. Their fit of Φ_{α} is given in Appendix A. When calculating Φ_{α} , the authors adopted the standard approximation in which the dependence of the gap on the absolute value of particle momentum is neglected (consequently, in the isotropic case the gap is a function of temperature only), $\Delta_{\mathbf{p}}^{(\alpha)} \approx \Delta^{(\alpha)}(|\mathbf{p}| = p_{F\alpha}) \equiv \Delta^{(\alpha)}(T)$. In this paper we also use this approximation.

Now, writing Eq. (38) for neutrons ($\alpha = n$) and for protons ($\alpha = p$), taking into account Eqs. (34) and (40)–(41), and equating prefactors at the same \mathbf{Q}_{α} in its left- and right-hand sides, we arrive at the set of linear equations for the matrix $\gamma_{\alpha \alpha'}$. Its solution has the form:

$$\gamma_{\alpha\alpha}(p_{F\alpha}) = \frac{m_\alpha}{m_\alpha^*} \frac{1}{S} \left\{ \left(1 + \frac{F_1^{\alpha\alpha}}{3}\right) \left(1 + \frac{F_1^{\beta\beta}}{3} \Phi_\beta\right) - \left(\frac{F_1^{\alpha\beta}}{3}\right)^2 \Phi_\beta \right\}, \quad (43)$$

$$\gamma_{\alpha\beta}(p_{F\alpha}) = \frac{1}{3} \frac{m_\beta}{\sqrt{m_\alpha^* m_\beta^*}} \frac{1}{S} \left(\frac{p_{F\beta}}{p_{F\alpha}}\right)^{3/2} F_1^{\alpha\beta} (1 - \Phi_\beta), \quad (44)$$

$$S \equiv \left(1 + \frac{F_1^{\alpha\alpha}}{3} \Phi_\alpha\right) \left(1 + \frac{F_1^{\beta\beta}}{3} \Phi_\beta\right) - \left(\frac{F_1^{\alpha\beta}}{3}\right)^2 \Phi_\alpha \Phi_\beta, \quad (45)$$

$$F_1^{\alpha\beta} \equiv f_1^{\alpha\beta} \sqrt{N_{0\alpha} N_{0\beta}}. \quad (46)$$

Here $\alpha \neq \beta$ (thus if, e.g., $\alpha = n$ then $\beta = p$). The effective masses m_α^* are related to the parameters $f_1^{\alpha\alpha'}$ through the equation (see Refs. [17,18]):

$$\frac{m_\alpha^*}{m_\alpha} = 1 + \frac{N_{0\alpha}}{3} \left[f_1^{\alpha\alpha} + \frac{m_\beta}{m_\alpha} \left(\frac{p_{F\beta}}{p_{F\alpha}}\right)^2 f_1^{\alpha\beta} \right], \quad \alpha \neq \beta. \quad (47)$$

4 The entrainment matrix

4.1 Superfluid current and the matrix $\rho_{\alpha\alpha'}$ in different limiting cases

Eqs. (33) and (36)–(37) allow one to calculate the entrainment matrix $\rho_{\alpha\alpha'}$ and to express it in terms of $\gamma_{\alpha\alpha'}(p_{F\alpha})$. This can be done, for instance, with the aid of Eq. (3) and the energy density given by Eq. (25). It is easier, however, to obtain $\rho_{\alpha\alpha'}$ by calculating the mass current density \mathbf{J}_α of quasiparticles (we remind that $\mathbf{V}_{qp} = 0$ in a chosen coordinate frame). When doing so we take account of the fact that the expression for the mass current density of non-superfluid Fermi-liquid can be applied to the superfluid state as well (see, e.g., Refs. [22,18]). In our case this means that the expression for \mathbf{J}_α has the form (see, e.g., Refs. [14,27]):

$$\mathbf{J}_\alpha = \sum_{\mathbf{p}\sigma} m_\alpha \frac{\partial H_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}}{\partial \mathbf{p}} \mathcal{N}_{\mathbf{p}+\mathbf{Q}_\alpha}^{(\alpha)}. \quad (48)$$

Substituting Eqs. (33) and (37) into Eq. (48) and performing a simple integration, we obtain the formulas, similar to Eqs. (1)–(2), with the entrainment matrix equal to $(\alpha, \alpha' = n, p)$:

$$\rho_{\alpha\alpha'} = \rho_\alpha \gamma_{\alpha\alpha'}(p_{F\alpha}) (1 - \Phi_\alpha). \quad (49)$$

From Eq. (44) it follows that the entrainment matrix is indeed symmetric in accordance with Eq. (3), $\rho_{\alpha\alpha'} = \rho_{\alpha'\alpha}$. At $T = T_{c\alpha}$ (where $T_{c\alpha}$ is the critical

temperature of quasiparticle species α) one has $\Phi_\alpha = 1$, and consequently, $\rho_{\alpha\alpha'} = 0$ and $\mathbf{J}_\alpha = 0$.

Let us analyze the matrix $\rho_{\alpha\alpha'}$ in different limiting cases.

I. Let $F_1^{\text{np}} = 0$, i.e., the interaction between neutrons and protons is absent. Then each component α of the mixture can be treated as an independent superfluid Fermi-liquid while the matrix $\rho_{\alpha\alpha'}$ is diagonal. Eqs. (49) and (43)–(46) yield

$$\rho_{\alpha\alpha} = \frac{\rho_\alpha (1 - \Phi_\alpha)}{1 + \Phi_\alpha F_1^{\alpha\alpha}/3}, \quad \rho_{\text{np}} = \rho_{\text{pn}} = 0. \quad (50)$$

The diagonal elements $\rho_{\alpha\alpha}$ coincide with the well-known expression for the superfluid density of one component Fermi-liquid (see, e.g., Refs. [22,28]).

II. Let the temperature of the mixture be equal to zero. Then $\Phi_\alpha = 0$ and the entrainment matrix can be rewritten in the form

$$\rho_{\alpha\alpha} = \rho_\alpha \frac{m_\alpha}{m_\alpha^*} \left(1 + \frac{F_1^{\alpha\alpha}}{3} \right), \quad (51)$$

$$\rho_{\text{np}} = \rho_{\text{pn}} = \frac{p_{\text{Fn}}^2 p_{\text{Fp}}^2}{9 \pi^4} m_{\text{n}} m_{\text{p}} f_1^{\text{np}}, \quad (52)$$

in agreement with the results of Borumand et al. [18].

III. Finally, let us suppose that the only one component α of the mixture is superfluid. In this case $\Phi_\beta = 1$ ($\beta \neq \alpha$) and we have:

$$\rho_{\alpha\alpha} = \rho_\alpha (1 - \Phi_\alpha) \frac{m_\alpha}{m_\alpha^*} \frac{(1 + F_1^{\alpha\alpha}/3) (1 + F_1^{\beta\beta}/3) - (F_1^{\alpha\beta}/3)^2}{(1 + \Phi_\alpha F_1^{\alpha\alpha}/3) (1 + F_1^{\beta\beta}/3) - (F_1^{\alpha\beta}/3)^2 \Phi_\alpha}, \quad (53)$$

$$\rho_{\beta\beta} = \rho_{\text{np}} = \rho_{\text{pn}} = 0. \quad (54)$$

We have calculated the entrainment matrix of a neutron-proton mixture at non-zero temperatures assuming singlet-state pairing of nucleons of both species and isotropic gaps. However, in reality neutron pairing occurs in the triplet 3P_2 state of nucleon pair with an anisotropic gap. In this case the expansion (34) becomes invalid because the momentum \mathbf{p} is no longer the only vector which characterizes the system in the absence of superfluid currents; the quantization axis specifies an additional direction. Therefore, Eq. (34) should be replaced by:

$$\Delta H_{\mathbf{p}}^{(\alpha)} = \sum_{\alpha'} \frac{\mathbf{G}_{\alpha\alpha'} \mathbf{Q}_{\alpha'}}{m_{\alpha'}}, \quad (55)$$

where $\mathbf{G}_{\alpha\alpha'}$ is a matrix composed of vectors. It can be, in principle, obtained from Eq. (32) in a manner similar to the derivation of the matrix $\gamma_{\alpha\alpha'}$. As

follows from Eq. (48), components of the entrainment matrix $\rho_{\alpha\alpha'}$ will be tensors (rather than scalars as in the isotropic case). These tensors will be expressed in terms of the Landau parameters $F_l^{\alpha\alpha'}$, with $l \geq 1$. The quantities $F_l^{\alpha\alpha'}$ at $l \geq 2$ are not known for neutron star matter and are not necessarily small. Thus, a strict calculation of the entrainment matrix in the case of triplet-state neutron pairing is rather complicated (the analogous problem in deriving the superfluid density was discussed in details by Leggett [23] in the context of the anisotropic phase of helium-3).

To proceed further we follow Baiko et al. [29] and assume that the neutron star matter can be treated as a collection of microscopic domains with arbitrary orientations of the quantization axis. Then the entrainment matrix, being averaged over the domains, will be “isotropic” (its elements will be scalars). Thus, one can use Eq. (49) for the averaged entrainment matrix by introducing an *effective* isotropic gap which we choose according to Baiko et al. [29] as

$$\Delta_{\text{eff}}^{(n)}(T) = \min \left\{ \Delta^{(n)}(|\mathbf{p}| = p_{\text{Fn}}) \right\}. \quad (56)$$

Here $\Delta_{\text{eff}}^{(n)}(T)$ is obtained as the minimum of the angle-dependent gap $\Delta_{\mathbf{p}}^{(n)}$ on the neutron Fermi surface. The use of Eq. (49) with the effective gap $\Delta_{\text{eff}}^{(n)}(T)$ allows one to obtain qualitatively correct results for the matrix $\rho_{\alpha\alpha'}$ in the case of triplet-state neutron pairing. The fit of $\Delta_{\text{eff}}^{(n)}(T)$ for the case of 3P_2 neutron pairing with $m_J = 0$ was obtained by Yakovlev and Levenfish [30] and is given in Appendix A.

4.2 Landau parameters

In order to find the matrix $\rho_{\alpha\alpha'}$ it is necessary to know the Landau parameters $F_1^{\alpha\alpha'}$ (and hence the quantities $f_1^{\alpha\alpha'}$, see Eq. (46)) for an asymmetric nuclear matter. In general, the quantity $f_l^{\alpha\alpha'}$ is a function of the baryon number density n_b and the asymmetry parameter $\delta = (n_n - n_p)/n_b$. Due to the charge symmetry of strong interactions one can write

$$f_l^{\text{nn}}(n_b, \delta) = f_l^{\text{pp}}(n_b, -\delta), \quad (57)$$

$$f_l^{\text{np}}(n_b, \delta) = f_l^{\text{np}}(n_b, -\delta). \quad (58)$$

The function $f_1^{\alpha\alpha'}(n_b, \delta)$ can be expanded in powers of $\delta \leq 1$. Neglecting all terms quadratic and higher order in δ , Eqs. (57)–(58) yield

$$f_1^{\text{nn}} = a(n_b) + \delta b(n_b) + O(\delta^2), \quad (59)$$

$$f_1^{\text{pp}} = a(n_b) - \delta b(n_b) + O(\delta^2), \quad (60)$$

$$f_1^{\text{np}} = c(n_b) + O(\delta^2). \quad (61)$$

This approximation was proposed by Haensel [31]. To calculate the functions $a(n_b)$, $b(n_b)$ and $c(n_b)$ we need to know the dependence $f_1^{\alpha\alpha'}(n_b)$ for any two values of the asymmetry parameter $\delta = \delta_1$ and $\delta = \delta_2$ (while the knowledge of this dependence for more than two values of δ would enable one to find the terms non-linear in δ in the expansions (59)–(61); to our best knowledge, these data are unavailable in the literature). The Landau parameters for asymmetric nuclear matter can be calculated microscopically within a nuclear many-body theory, starting from the nucleon-nucleon interaction in vacuum. However, nearly all existing calculations are limited to a simpler case of a pure neutron matter or symmetric nuclear matter. In the case of pure neutron matter calculations of the $l = 1$ Landau parameters were performed in some range of neutron matter density above nuclear density (see Refs. [32]–[36]). In the case of symmetric nuclear matter, calculations were usually done at normal nuclear density (see Refs. [37]–[39]) with a notable exception of the calculation of Jackson et al. [33]. We are aware of only one calculations of Landau parameters in asymmetric nuclear matter in beta equilibrium with electron gas (simplest model of neutron star matter), by Shen et al. [40], who however restricted to $l = 0$ parameters, while we need $l = 1$ ones. In view of this situation, we decided to use the density dependent $l = 1$ parameters for symmetric nuclear and pure neutron matter calculated by Jackson et al. [33] more than two decades ago.

These authors calculated the Landau parameters for symmetric nuclear matter ($\delta = 0$) and for pure neutron matter ($\delta = 1$) using two model potentials of nucleon-nucleon interaction: Bethe-Johnson v6 (BJ v6) and Reid v6. In the case of the symmetric nuclear matter the quantities $f_1^{\alpha\alpha'}$ can be expressed as

$$f_1^{\text{nn}}(n_b, 0) = f_1^{\text{pp}}(n_b, 0) = \frac{F_1(n_b, 0) + F_1'(n_b, 0)}{2N_{0\text{sym}}}, \quad (62)$$

$$f_1^{\text{np}}(n_b, 0) = \frac{F_1(n_b, 0) - F_1'(n_b, 0)}{2N_{0\text{sym}}}. \quad (63)$$

Here $N_{0\text{sym}} \equiv N_{0n}(n_b, 0) = N_{0p}(n_b, 0)$. The plots of F_1 and F_1' versus the wave number $k_{\text{Fsym}} = (3\pi^2 n_b/2)^{1/3}$ are given in Figures 14 and 17 of Ref. [33] for the Reid v6 and BJ v6 interactions, respectively. We have fitted these functions by simple analytical formulas which are given in Appendix B. Eqs. (59) – (61) yield

$$a(n_b) = f_1^{\text{nn}}(n_b, 0) = f_1^{\text{pp}}(n_b, 0), \quad c(n_b) = f_1^{\text{np}}(n_b, 0). \quad (64)$$

Now, considering the pure neutron matter, we obtain

$$f_1^{\text{nn}}(n_b, 1) = \frac{F_1(n_b, 1)}{N_{0n}(n_b, 1)}. \quad (65)$$

The function $F_1(n_b, 1) \equiv F_1(k_{\text{Fpure}})$, with $k_{\text{Fpure}} = (3\pi^2 n_b)^{1/3}$, is plotted in Figures 20 and 22 of Ref. [33] for the Reid v6 and BJ v6 interactions, respec-

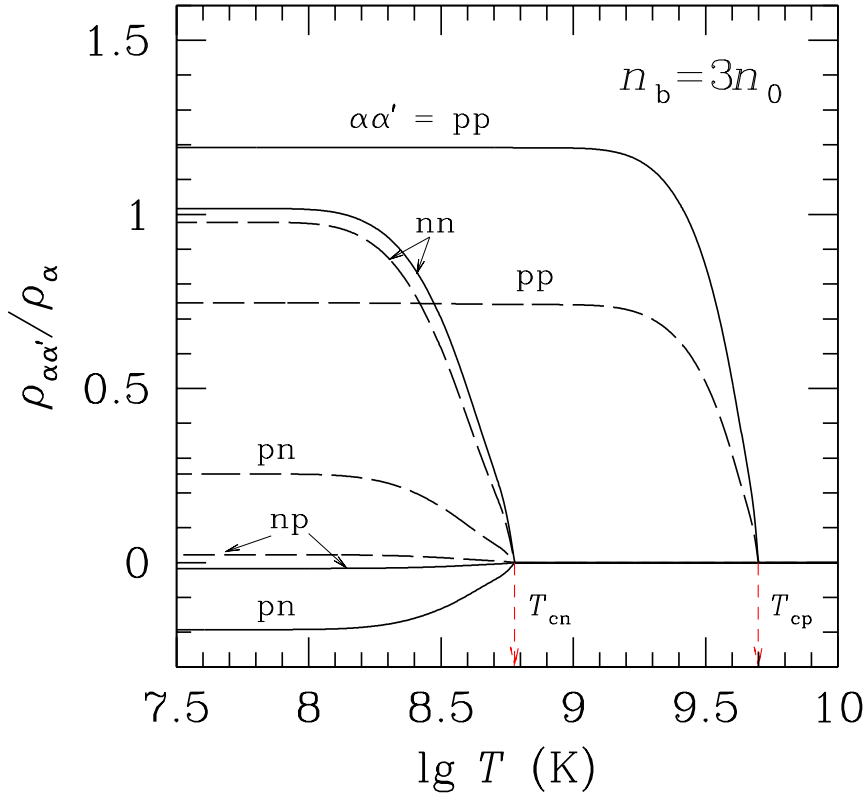


Fig. 1. The quantities $\rho_{\alpha\alpha'}/\rho_\alpha$ ($\alpha, \alpha' = n$ or p) versus temperature T for two interactions: BJ v6 (solid lines) and Reid v6 (long dashes) at the baryon number density $n_b = 3n_0$. The proton ($T_{cp} = 5 \times 10^9$ K) and the neutron ($T_{cn} = 6 \times 10^8$ K) critical temperatures are marked by the vertical dashed arrows.

tively. The fitting formulas for $F_1(k_{F\text{pure}})$ are given in Appendix B. From Eq. (59) we have

$$b(n_b) = f_1^{\text{nn}}(n_b, 1) - a(n_b). \quad (66)$$

Thus, assuming that $f_1^{\alpha\alpha'}$ is linear in δ , one can calculate the Landau parameters of the asymmetric nuclear matter and find the entrainment matrix.

Our results are illustrated in Fig. 1. We show the temperature dependence of $\rho_{\alpha\alpha'}/\rho_\alpha$ ($\alpha, \alpha' = n$ or p) for the BJ v6 (solid lines) and Reid v6 (long dashes) interactions. The number density of baryons is taken to be $n_b = 3n_0$, where $n_0 = 0.16 \text{ fm}^{-3}$ is the number density of nuclear matter at saturation. For the chosen n_b , the equation of state of Heiselberg and Hjorth-Jensen [41] yields $\delta = 0.837$. The nucleon critical temperatures are taken to be $T_{cn} = 6 \times 10^8$ K, $T_{cp} = 5 \times 10^9$ K. This information is sufficient for computing the entrainment matrix at any T . However, it should be stressed, that our approach is not self-consistent. Strictly speaking, one needs to calculate the equation of state, the Landau parameters, and the nucleon critical temperatures using *one model of strong interactions*.

At $T \geq T_{cp}$ in Fig. 1 the matter is non-superfluid and all the matrix elements $\rho_{\alpha\alpha'} = 0$. At $T_{cn} \leq T \leq T_{cp}$ protons become superfluid and hence $\rho_{pp} \neq 0$. Finally, at $T < T_{cn}$ both protons and neutrons are superfluid and all the matrix elements are non-zero. With decreasing temperature, the elements $\rho_{\alpha\alpha'}$ rapidly approach their asymptotes (51) and (52). Notice, that ρ_{pp}/ρ_p depends essentially on the model of strong interactions employed (the difference between the solid and dashed curves marked by pp is large), for ρ_{pn}/ρ_p and ρ_{np}/ρ_n the dependence is even stronger: even the signs of these quantities differs for the BJ v6 and Reid v6 interactions (for the Reid v6 interaction $\rho_{pn} = \rho_{np} < 0$ because $f_1^{np} < 0$; see Eq. (52)). The quantity ρ_{pp}/ρ_p at $T \rightarrow 0$ is equal to $\rho_{pp}/\rho_p \approx 1.19$ for the BJ v6 interaction and to $\rho_{pp}/\rho_p \approx 0.75$ for the Reid v6 interaction. Thus, the value of ρ_{pp}/ρ_p is approximately half of that obtained in the estimate of Borumand et al. [18], $\rho_{pp}/\rho_p \approx 2$.

5 Conclusions

We have derived the entrainment matrix $\rho_{\alpha\alpha'}$ at non-zero temperatures. The calculation is performed in the frame of the Landau Fermi-liquid theory generalized by Larkin and Migdal [21] and Leggett [22] to the case of superfluid matter. The expressions for $\rho_{\alpha\alpha'}$ reproduce two limiting cases studied in the literature: the case of zero temperature and the case in which the interaction between neutrons and protons is absent.

The results are presented in the form convenient for their practical use. In particular, for calculating the entrainment matrix one needs the Landau parameters of symmetric nuclear and pure neutron matter. These parameters were taken from the paper by Jackson et al. [33] for two model potentials of nucleon-nucleon interaction (BJ v6 and Reid v6) and approximated by simple analytical formulas. While results for ρ_{nn} are quite similar for both nucleon-nucleon potentials, the values of ρ_{pp} differ by some thirty percent, while much smaller non-diagonal matrix elements, ρ_{np} and ρ_{pn} differ in sign. Clearly, results for the non-diagonal entrainment matrix elements are very sensitive to the nucleon-nucleon interaction. Moreover, as we used old values of the Landau parameters, our results should be updated as soon as new, more realistic values of the $l = 1$ parameters become available, obtained, e.g., via modern renormalization group approach (see, e.g., Ref. [36]) applied to pure neutron matter, symmetric nuclear matter, and maybe also to the most astrophysically interesting case of asymmetric nuclear matter.

The generalization of the entrainment matrix to the case of three or more interacting baryon species is straightforward. It just implies an increased corresponding number of baryon indices in all formulas and an associated increase of the dimension of matrices $\boldsymbol{\rho}$ and $\boldsymbol{\gamma}$.

The *temperature dependence* of the matrix $\rho_{\alpha\alpha'}$, fully described by the universal function $\Phi_\alpha(T)$, is known rather reliably. Unfortunately, this cannot be said about the values of the Landau parameters. Clearly, new calculations of the Landau parameters of asymmetric nuclear matter would be highly desirable.

The matrix $\rho_{\alpha\alpha'}$ at non-zero temperatures is needed to study the kinetics of the neutron star matter as well as to investigate the dynamical evolution of neutron stars, especially, their pulsations. As we have shown, the entrainment matrix varies with temperature. Because the matrix $\rho_{\alpha\alpha'}$ enters hydrodynamic equations which determine the neutron star pulsations, the frequencies of (superfluid) pulsation modes should vary with T and hence (due to a star cooling or heating), with time. This gives a potentially powerful method to probe very subtle properties of superdense matter by measuring the dependence of pulsation frequencies on time. We intend to consider the related problems in a separate publication.

Acknowledgements

The authors are grateful to D.G. Yakovlev for discussion. One of the authors (M. Gusakov) also acknowledges the excellent working condition at the N. Copernicus Astronomical Center in Warsaw, where this study was completed.

This research was supported by RFBR (grants 03-07-90200 and 05-02-16245), the Russian Leading Science School (grant 1115.2003.2), INTAS YSF (grant 03-55-2397), the Russian Science Support Foundation, and by the Polish MNiI Grant No. 1 P03D 008 27.

A The analytical approximation of the function Φ_α

Eq. (42) for Φ_α can be written as:

$$\Phi_\alpha = 2 \int_0^\infty dx \frac{\exp(\sqrt{x^2 + v^2})}{[\exp(\sqrt{x^2 + v^2}) + 1]^2}. \quad (\text{A.1})$$

Here $v = \Delta(T)/T$; $\Delta(T)$ is the gap which depends on the type of pairing. For singlet-state pairing of nucleon species α the function v can be taken from Levenfish and Yakovlev [42]:

$$v = \frac{\Delta^{(\alpha)}(T)}{T} = \sqrt{1 - \tau} \left(1.456 - \frac{0.157}{\sqrt{\tau}} + \frac{1.764}{\tau} \right), \quad \tau = \frac{T}{T_{c\alpha}}. \quad (\text{A.2})$$

In the case of triplet-state neutron pairing the effective gap should be introduced which is defined by Eq. (56). The function v is now given by (see Ref. [30]):

$$v = \frac{\Delta_{\text{eff}}^{(n)}(T)}{T} = \sqrt{1 - \tau} \left(0.7893 + \frac{1.188}{\tau} \right), \quad \tau = \frac{T}{T_{\text{cn}}}. \quad (\text{A.3})$$

Gnedin and Yakovlev [26] calculated the function $\Phi_\alpha(v)$ in a wide range of v (for the problem of thermal conductivity). These authors fitted $\Phi_\alpha(v)$ by a simple analytical formula which is correct at any v and satisfies the asymptotes $\Phi_\alpha = \sqrt{2\pi v} e^{-v}$ at $v \rightarrow +\infty$:

$$\Phi_\alpha = \left[0.9443 + \sqrt{(0.0557)^2 + (0.1886v)^2} \right]^{1/2} \exp \left(1.753 - \sqrt{(1.753)^2 + v^2} \right). \quad (\text{A.4})$$

Calculation and fit errors do not exceed 2.6%.

B Fits to F_1 and F'_1

B.1 Symmetric nuclear matter

In the symmetric nuclear matter $\delta = 0$. The plots of functions F_1 and F'_1 versus the wave number $k_{\text{Fsym}} = (3\pi^2 n_b/2)^{1/3}$ are given by Jackson et al. [33] for the model interactions BJ v6 and Reid v6 (see Section 4.2 for details). We have approximated these functions by simple analytical formulas in the interval $1.2 \text{ fm}^{-1} \leq k_{\text{Fsym}} \leq 2.0 \text{ fm}^{-1}$. For the BJ v6 interaction we obtain

$$F_1(n_b, 0) = -0.6854 + 0.6724 k_{\text{Fsym}} - 0.5180 (k_{\text{Fsym}})^2, \quad (\text{B.1})$$

$$F'_1(n_b, 0) = 1.723 - 1.520 k_{\text{Fsym}} + 0.03498 (k_{\text{Fsym}})^2, \quad (\text{B.2})$$

while for the Reid v6 interaction

$$F_1(n_b, 0) = 1.034 - 1.866 k_{\text{Fsym}} + 0.5455 (k_{\text{Fsym}})^2, \quad (\text{B.3})$$

$$F'_1(n_b, 0) = 0.6973 + 0.1403 k_{\text{Fsym}} - 0.5303 (k_{\text{Fsym}})^2. \quad (\text{B.4})$$

B.2 Pure neutron matter

In the pure neutron matter $\delta = 1$. The plots of the function F_1 versus the wave number of the pure neutron matter $k_{\text{Fpure}} = (3\pi^2 n_b)^{1/3}$ are given by Jackson et al. [33] (see also Section 4.2).

The fit for the BJ v6 interaction has the form

$$F_1(n_b, 1) = 0.1473 + 0.7372 k_{\text{Fpure}} - 1.0414 (k_{\text{Fpure}})^2 + 0.1958 (k_{\text{Fpure}})^3, \quad (\text{B.5})$$

while for the Reid v6 we obtain

$$F_1(n_b, 1) = -0.2729 + 1.5545 k_{\text{Fpure}} - 1.3225 (k_{\text{Fpure}})^2 + 0.2393 (k_{\text{Fpure}})^3. \quad (\text{B.6})$$

The fitting formulas (B.5) and (B.6) correctly describe the function $F_1(n_b, 1)$ in the interval $0.75 \text{ fm}^{-1} \leq k_{\text{Fpure}} \leq 3.0 \text{ fm}^{-1}$.

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